APPLICATIONS TO EIGENFUNCTIONS OF THE LAPLACIAN ON THE TORUS
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0.1. **Eigenfunctions of the Laplacian.** Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected, planar domain, with piecewise smooth boundary. The Dirichlet Laplacian is (the self-adjoint extension of) the operator$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ acting on functions $f \in C_c^\infty(\Omega \backslash \partial\Omega)$ which vanish in a neighborhood of the boundary $\partial\Omega$. It is known that there is an ONB of $L^2(\Omega)$ consisting of eigenfunctions of Δ: $-\Delta f_n = E f_n$, that the eigenvalues cluster only at infinity: $E_n \to \infty$.

0.2. **Examples.**

0.2.1. *Eigenfunctions on an interval.* We take B to be an interval $B = [0, a]$ of length a. Then the functions

$$f_n(x) := \sqrt{\frac{2}{a}} \sin \frac{\pi n x}{a}, \qquad n \geq 1$$

vanish at the boundary points 0, a, are orthonormal on $[0, a]$, and are eigenfunctions of the Laplacian $\Delta = \frac{\partial^2}{\partial x^2}$ with eigenvalue $E_n = \pi^2 n^2$.

0.2.2. *Eigenfunctions on the circle.* Take $S^1 = \mathbb{R}/\mathbb{Z}$ the unit circle. Then an ONB of eigenfunctions are the elementary exponentials $e_n(x) = e^{2\pi i n x}$, $n \in \mathbb{Z}$, with eigenvalue $4\pi^2 n^2$, which appears with multiplicity two (except when $n = 0$).

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0.2.3. *Eigenfunctions on the rectangle billiard.* We take a rectangle $R = [0,a] \times [0,b]$ with side-lengths a and b. The clearly the functions

$$f_{m,n}(x,y) = \frac{2}{\sqrt{ab}} \sin \frac{\pi m x}{a} \sin \frac{\pi n y}{b}$$

vanish on the boundary $\partial R\{x = 0, a\} \cup \{y = 0, b\}$, and are eigenfunctions of the Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ with eigenvalue

$$E_{m,n} = \pi^2 \Big((\frac{m}{a})^2 + (\frac{n}{b})^2 \Big).$$

They give an orthonormal basis (ONB) of all Dirichlet eigenfunctions on the rectangle R.

0.2.4. *Toral eigenfunctions:* Any eigenfunction on the torus $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with eigenvalue $4\pi^2 E$ is of the form

$$f(x) = \sum_{\substack{\mu \in \mathbb{Z}^2 \\ |\mu|^2 = E}} a(\mu) e_\mu(x), \qquad e_\mu(x) := e^{2\pi i \langle \mu, x \rangle}.$$

The L^2 norm is

$$||f||_2^2 = \int_{\mathbf{T}^2} |f(x)|^2 dx = \sum_\mu |a(\mu)|^2$$

0.2.5. *The disk.* Take $\Omega = \{x^2 + y^2 \le 1\}$ the unit disk. The eigenfunctions are, in polar coordinates (r, θ),

$$f_{n,k,\pm}(r,\theta) = J_n(j_{n,k} r) \begin{cases} \sin(n\theta) \\ \cos(n\theta) \end{cases}$$

where $J_n(u)$ is the n-the Bessel function, and $j_{n,k}$ is the k-th zero of J_n. These are eigenfunctions with eigenvalue $j_{n,k}^2$. The spectrum has multiplicity 2.

Recall: The Bessel function $J_n(x)$ is a solution of the ODE (Bessel's equation)

$$x^2 f'' + x f' + (x^2 - n^2) f = 0$$

which is finite at $x = 0$. It has a power series expansion

$$J_n(x) = \sum_{m=0}^\infty \frac{(-1)^m}{m!(m+n)!} \Big(\frac{z}{2}\Big)^{2m+n}$$

and admits an integral representation

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(n\tau - x \sin \tau) d\tau = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n\tau - x\sin\tau)} d\tau$$

0.3. **Hearing the shape of a drum.** The question "can we hear the sound of a drum" is the question of what we can recover about the geometry of Ω from the spectrum $\{E_n\}$.

Weyl's law (1911) says that we can recover the *area* of the drum:

$$N(X) := \#\{n \ge 1 : E_n \le X\} \sim \frac{\text{area}(\Omega)}{4\pi} X$$

More generally, for bounded domains in $\Omega \subset \mathbb{R}^d$ with nice boundary, we have

$$N(X) = \#\{n \ge 1 : E_n \le X\} \sim \frac{\omega_d}{(2\pi)^d} \text{vol}(\Omega) X$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

Example: Weyl's law for a rectangle reduces to our asymptotics of the number of lattice points in a quarter-ellipse:

$$\#\{E_{m,n} \leq X\} = \#\{(m,n) : m,n \geq 1, (\frac{m}{a})^2 + (\frac{n}{b})^2 \leq X/\pi^2\}$$

which we know is asymptotically the area of the quarter-ellipse, namely

$$\frac{1}{4}\operatorname{area}\{(x/a)^2 + (y/b)^2 \leq 1\}\frac{X}{\pi^2} = \frac{1}{4}\pi ab\frac{X}{\pi^2} = \frac{ab}{4\pi}X = \frac{\operatorname{area}(R)}{4\pi}X$$

which is exactly the statement of Weyl's law!

0.3.1. *Using the heat kernel.* For two-dimensional drums with smooth boundary $\partial\Omega$, we can also hear the *length* of the boundary, and the connectivity (number of holes) $h(\Omega)$. The device to do this is not the asymptotics of the spectral staircase $N(X)$, but rather the small time asymptotics of the heat kernel $\sum_{n\geq 1} e^{-E_n t}$: As $t \searrow 0$,

$$\sum_{n\geq 1} e^{-E_n t} \sim c_1 \frac{\operatorname{area}(\Omega)}{t} - c_2 \frac{\operatorname{length}\partial\Omega}{\sqrt{t}} + \frac{1-h(\Omega)}{6} + o(1)$$

with $c_1 = 1/4\pi$, $c_2 = 1/8\sqrt{\pi}$.

For a closed compact smooth surface (no boundary), the small time asymptotics of the trace of the heat kernel is given by

$$\sum_{n\geq 0} e^{-tE_n} = \frac{\operatorname{area}(M)}{4\pi t} + \frac{1-g(M)}{6} + O(t), \qquad t \searrow 0,$$

where $g(M)$ is the genus of M.

0.4. **The heat kernel on the interval and the Riemann zeta function.** We saw that the eigenvalues of the Laplacian d^2/dx^2 on the interval $\Omega = [0, A]$ are $E_n = (\pi n/A)^2$, $n = 1, 2, \dots$. Let's use this information to compute the small time asymptotics of the trace of the heat kernel $K_\Omega(t)$ in this case:

$$K_\Omega(t) := \sum_{n\geq 1} e^{-tE_n}$$

Setting

$$\tau := t\pi/A^2$$

we have

$$K_\Omega(t) = \sum_{n\geq 1} e^{-\pi\tau n^2} = \frac{\theta(\tau) - 1}{2}$$

where

$$\theta(\tau) = \sum_{m\in\mathbb{Z}} e^{-\pi\tau m^2}.$$

Proposition 0.1. *We have a functional equation*

$$\theta(\frac{1}{\tau}) = \sqrt{\tau}\theta(\tau)$$

Proof. Let $g(x) = e^{-\pi x^2}$, and $g_\tau(x) := g(\sqrt{\tau}x) = e^{-\pi\tau x^2}$. As we saw in the homework exercises, g is it's own Fourier transform:

$$\widehat{g}(y) = g(y).$$

Hence by the properties of the Fourier transform under dilation,

$$\widehat{g}_\tau(y) = \frac{1}{\sqrt{\tau}}\widehat{g}(\frac{y}{\sqrt{\tau}}) = \frac{1}{\sqrt{\tau}}e^{-\pi y^2/\tau}.$$

Using the Poisson sumation formula we find

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} g_\tau(n) = \sum_{m\in\mathbb{Z}} \widehat{g}_\tau(m) = \frac{1}{\sqrt{\tau}} \sum_{m\in\mathbb{Z}} e^{-\pi m^2/\tau} = \frac{1}{\sqrt{\tau}}\theta(\frac{1}{\tau}).$$

□

Corollary 0.2.

$$\theta(\tau) \sim \frac{1}{\sqrt{\tau}}\Big(1 + O(e^{-\pi/\tau})\Big), \qquad \tau \searrow 0$$

Proof. We clearly have

$$\theta(\tau) = 1 + O(e^{-\pi\tau}), \qquad \tau \to +\infty$$

and hence when $\tau \searrow 0$, so that $1/\tau \to +\infty$,

$$\theta(\tau) = \frac{1}{\sqrt{\tau}}\theta(\frac{1}{\tau}) = \frac{1}{\sqrt{\tau}}\Big(1 + O(e^{-\pi/\tau})\Big)$$

□

Corollary 0.3. *For the interval* $\Omega = [0, A]$,

$$K_\Omega(t) = \frac{\operatorname{length}(\Omega)}{2\sqrt{\pi}}\frac{1}{\sqrt{t}} - \frac{1}{2} + o(1), \qquad \tau \to 0$$

Proof. We use $K(t) = (\theta(\tau) - 1)/2$, $\tau = t\pi/A^2$. Then

$$K(t) = -\frac{1}{2} + \frac{1}{2}\theta(\tau) = -\frac{1}{2} + \frac{1}{2}\Big(\frac{1}{\sqrt{\tau}} + o(1)\Big) = -\frac{1}{2} + \frac{1}{2}\frac{A}{\sqrt{\pi t}} + o(1).$$

□

We can now compute the small time asymptotics for the heat kernel of the rectangle R of side-lengths A and B.

Exercise 1. *As* $t \searrow 0$,

$$K_R(t) = \frac{\operatorname{area}\Omega}{4\pi t} - \frac{\operatorname{length}\partial\Omega}{8\sqrt{\pi t}} + \frac{1}{4} + o(1).$$

0.5. **Nodal lines.** Nodes are locations on a plucked string which do not move. In vibration of a surface or membrane, the nodes become nodal lines, lines on the surface where the surface is motionless, dividing the surface into separate regions vibrating with opposite phase. The deflection $u(x, t)$ of the ideal string/membrane satisfies the wave equation $\partial^2 u/\partial^2 t = c^2\Delta u$. Separation of variables $u(x, t) = T(t)\phi(x)$ implies ϕ is an eigenfunction of the Laplacian: $-\Delta\phi = E\phi$. The nodes are just the zeros of ϕ.

Let $\Omega \subset \mathbb{R}^2$ be a nice planar domain. For an eigenfunction f_E, the nodal line (nodal set) is $Z_f := \{x \in \Omega : f(x) = 0\}$. The nodal domains are the connected components of the complement $\Omega\backslash Z_f$ of the nodal set.

Theorem 0.4 (Courant's nodal line theorem (1923))**.** *The number ν_n of nodal domains of the n-th eigenfunction is at most $\nu_n \leq n$.*

This was sharpened by Pleijel (1956) to $\nu_n/n \leq 4/j_{0,1}^2 = 0.691\dots$

0.5.1. *The length of the nodal line:* We know that (when $\partial\Omega$ is real-analytic)

$$c\sqrt{E_n} \leq \operatorname{length}(\{f_n = 0\}) \leq C\sqrt{E_n}$$

It is an exercise to show that this holds for the rectangle.

0.6. **Persistent components of nodal lines on the torus.** As an application of our work on lattice points on circles, we can say something about nodal lines on the torus (or the square). The statement is that they usually have to *vary*!.

This is not strictly true. For instance, the line $y = 1/2$ is part of the nodal line for all eigenfunctions $\sin(\pi m x)\sin(\pi 2 n y)$.

It turns out that this is the only such curve which has this property. We show that if Σ is not part of a closed geodesic (i.e. line with rational slope), then any eigenfunction with sufficiently large eigenvalue cannot vanish identically on it. The main part is the case when Σ is curved:

Theorem 0.5 (Bourgain-Rudnick)**.** *Let $\Sigma \subset \mathbf{T}^2$ be real analytic and have nowhere zero curvature. Then there is some $E(\Sigma) > 0$ so that the curve Σ cannot be contained in the nodal line of any eigenfunction f_E on the torus with eigenvalue $E \geq E(\Sigma)$, that is there is some point $x \in \Sigma$ for which $f_E(x) \neq 0$.*

Proof. Let μ_0 be such that the Fourier coefficient $a(\mu_0)$ is maximal:

$$|a(\mu_0)| \geq |a(\mu)|, \qquad \forall \mu.$$

We may replace f by $f/a(\mu_0)$ so as to obtain

$$a(\mu_0) = 1 \geq |a(\mu)|, \qquad \forall \mu.$$

Consider the integral ("period") of f_E along the curve

$$J := \frac{1}{L}\int_\Sigma e_{-\mu_0} f_E = \frac{1}{L}\int_0^L f(\gamma(t)) e^{-2\pi i\langle \mu, \gamma(t)\rangle} dt$$

If the curve Σ lies in the nodal line of f_E, then $J = 0$. On the other hand, using the Fourier expansion we get

$$J = \sum_\mu a(\mu)\frac{1}{L}\int_\Sigma e_{\mu-\mu_0} =: \sum_\mu a(\mu) I(\mu - \mu_0).$$

where for $\xi \in \mathbb{R}^2$,

$$I(\xi) := \frac{1}{L}\int_0^L e^{i\langle \xi, \gamma(t)\rangle} dt$$

is an oscillatory integral.

The term $\mu = \mu_0$ gives a contribution of $a(\mu_0)\frac{1}{L}\int_\Sigma 1 = 1$. Therefore since $J = 0$ we obtain

$$-1 = \sum_{\mu \neq \mu_0} a(\mu) I(\mu - \mu_0)$$

Hence we obtain

$$1 \leq \sum_{\mu \neq \mu_0} |a(\mu)||I(\mu - \mu_0)|$$

We will use van der Corput's Lemmas to bound, using the nowhere vanishing curvature condition (and real analyticity to ensure at most finitely many critical points), see Lemma 0.6 below,

$$|I(\xi)| \ll |\xi|^{-1/2}.$$

By Jarnik's theorem, for all except perhaps one other frequency $\mu_1 \neq \mu_0$, we must have

$$|\mu - \mu_0| \gg E^{1/6}, \qquad \mu \neq \mu_0, \mu_1.$$

Since $|a(\mu)| \leq 1$, we obtain that

$$\Big|\sum_{\mu\neq\mu_0,\mu_1} a(\mu) I(\mu-\mu_0)\Big| \leq \sum_{\mu\neq\mu_0,\mu_1} |I(\mu-\mu_0)| \ll \sum_{\mu\neq\mu_0,\mu_1} \frac{1}{|\mu-\mu_0|^{1/2}}$$
$$\ll \#\{\mu\in\mathbb{Z}^2 : |\mu|^2 = E/4\pi^2\} E^{-1/12} = o(1)$$

because we saw that $\#\{\mu \in \mathbb{Z}^2 : |\mu|^2 = E/4\pi^2\} = O(E^{\varepsilon})$, for all $\varepsilon > 0$. So if there is no $\mu_1 \neq \mu_0$ which is close to μ_0 then we have a contradiction: $1 = o(1)$.

If such μ_1 exists, then note that $I(0) = 1$, and that $|I(\xi)| \leq 1$ for all $\xi \in \mathbb{R}^2$, and in fact

$$|I(\xi)| = 1 \longleftrightarrow \exists c \text{ s.t. } e^{i\langle \xi, \gamma(t)\rangle} = c, \quad \forall t \in [0, L]$$

is *constant*. Indeed, this follows from the case of equality in the triangle inequality, namely

$$\Big|\int_0^L e^{ig(t)} dt\Big| = \int_0^L |e^{ig(t)}| dt$$

if and only if there is some constant c of absolute value 1 so that $e^{ig(t)} = c$ for all $t \in [0, L]$.

Now the condition

$$e^{i\langle \xi, \gamma(t)\rangle} = c, \quad \forall t \in [0, L]$$

is equivalent to

$$\langle \xi, \gamma(t)\rangle = C$$

which means that $\gamma(t) = at + b$ is a line (the constant speed condition $|\dot\gamma| \equiv 1$ forces $a = \pm 1$). But then the curvature is identically zero, unlike our assumption that the curvature is never zero. So we obtain that

$$|I(\xi)| < 1, \quad \forall \xi \neq 0.$$

Since $I(\xi) \to 0$ as $|\xi| \to \infty$, we infer that there is some constant $c < 1$ strictly smaller than 1 so that

$$|I(\xi)| \leq c < 1, \qquad \forall |\xi| \geq 1.$$

Thus in particular $|I(\mu_1 - \mu_0)| \leq c < 1$ and so we obtain

$$1 \leq |a(\mu_1)| |I(\mu_1 - \mu_0)| + o(1) \leq 1 \cdot c + o(1) < 1$$

for $E \gg 1$. This gives the required contradiction. $\square$

0.7. **Bounding the oscillatory integral.**

0.7.1. *Background on curvature:* Suppose $\gamma : [0, L] \to \Sigma \subset \mathbb{R}^2$ is an arc-length parameterization of a plane curve Σ. The unit tangent to the curve at the points $\gamma(s)$ is $T(s) = \dot\gamma(s)$. Let $N(s)$ be the unit normal (a choice of sign needs to be made here) to the curve at the points $\gamma(s)$. The pair $(T(s), N(s))$ is the *moving frame*, giving an orthonormal pair of vectors at each point on the curve.

The curvature of the curve at the points $\gamma(s)$ is the instantaneous rate of change of the angle $\alpha(s)$ between the tangent to the curve and a fixed direction e, say the x-axis:

$$\kappa(s) = |\frac{d\alpha}{ds}|$$

An alternative description is

$$\kappa(s) = ||\dot T(s)|| = ||\ddot\gamma(s)||.$$

To see this, note that because the curvature is by arc-length means that $|\dot\gamma(s)| \equiv 1$. This condition forces $\ddot\gamma(s)$ to be orthogonal to $\dot\gamma(s) = T(s)$, and hence is a multiple of the unit normal $N(s)$. Indeed

$$0 = \frac{d}{ds}(||\dot\gamma(s)||^2) = 2\ddot\gamma(s) \cdot \dot\gamma(s)$$

so that $\dot\gamma(s) \cdot \ddot\gamma(s) = 0$. Hence

$$\ddot\gamma(s) = \pm\kappa(s)N(s)$$

for some $\kappa \geq 0$. Then κ is exactly the curvature (these are the Frenet-Serret equations for the plane). Indeed, expand the fixed direction vector e in terms of the moving frame:

$$e = \cos\alpha T + \sin\alpha N$$

so that $\cos\alpha = e \cdot T(s)$. Then

$$-\sin\alpha\frac{d\alpha}{ds} = \frac{d}{ds}\Big(e \cdot T(s)\Big) = e \cdot \frac{dT}{ds} = e \cdot \kappa N = \kappa\sin\alpha$$

and taking absolute values gives $\kappa = |\frac{d\alpha}{ds}|$.

0.7.2. *Bounding the oscillatory integral.* We defined for $0 \neq \xi \in \mathbb{R}^2$

$$I(\xi) = \frac{1}{L}\int_0^L e^{i\langle\xi,\gamma(t)\rangle}dt$$

Write $\xi = |\xi|u$, where $u = \xi/|\xi|$ is a unit vector, and set

$$\phi_u(t) = \langle u, \gamma(t)\rangle$$

so that we are dealing with (ignoring the factor of $1/L$) an oscillatory integral

$$I(\lambda, \phi_u) = \int_0^L e^{i\lambda\phi_u(t)}dt$$

Lemma 0.6. *Under the assumption that Σ is real analytic with nowhere zero curvature, we have*

$$I(\lambda, \phi_u) \ll_\Sigma \frac{1}{|\lambda|^{1/2}}$$

Proof. We want to use van der Corput's lemmas to bound the integral, and it is important that the estimates be uniform in $u \in S^1$. To do so, we have to investigate the critical points if the phase function $\pi_u(t)$.

We note that

$$\phi'(t) = \langle \dot\gamma(t), u\rangle = \langle T(t), u\rangle, \qquad \phi''(t) = \langle \ddot\gamma(t), u\rangle = \kappa(t)\langle N(t), u\rangle$$

on using the Frenet Serret equation $\cdot T = \kappa N$, where $T = \dot\gamma$ is the unit tangent to the curve, and N is the unit normal. Hence we have

$$(\phi')^2 + \frac{1}{\kappa^2}(\phi'')^2 = 1$$

In particular, at a critical point $\phi'(t_0) = 0$, we must have $|\phi''(t_0)| = 1$. Note that the critical points are precisely where u is orthogonal to the unit tangent T, that is points where the unit normal points in the direction $\pm u$. Since we assume that γ is real analytic, there are only finitely many such points.

Let $J_1, \dots, J_N$ be the union of intervals on which $|\phi''(t)| \geq \kappa(t)/2 \geq \kappa_{\min}/2$, where

$$\kappa_{\min} = \min_{[0,L]} \kappa(t).$$

These contain all critical points. By the second van der Corput Lemma, on each such interval we have a bound (independent of u!)

$$I(\lambda, \phi_u) \leq \frac{8}{\sqrt{\min(|\phi''|; t \in J_j)}} \frac{1}{|\lambda|^{1/2}} \leq \frac{8\sqrt{2}}{\sqrt{\kappa_{\min}}} \frac{1}{|\lambda|^{1/2}}$$

On the complement of these N intervals, which is a union of at most $N+1$ intervals $J'_1, \dots, J'_{N+1}$, we must have

$$(\phi')^2 = 1 - \frac{1}{\kappa^2}(\phi'')^2 \geq 1 - (\frac{1}{2})^2 = \frac{3}{4}$$

Hence we can use the first van der Corput lemma (we can guarantee monotonicity of ϕ' after a further subdivision into finitely many subintervals; again real analyticity is used here) to obtain that on each such subinterval J',

$$I(\phi_u, J'; \lambda) \leq \frac{4}{\min(|\phi'(t)| : t \in J')} \frac{1}{\lambda} \leq \frac{8}{\sqrt{3}} \frac{1}{\lambda}$$

□